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ON RENEWAL DECISIONS

by

C. DERMAN, G. J. LIEBERMAN and S. ROSS

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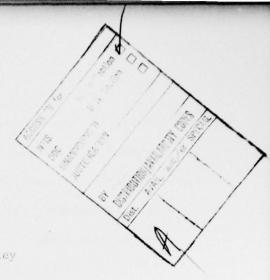DEPARTMENT OF STATISTICS
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ON RENEWAL DECISIONS[*]

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1. Statement of Problem

A system must operate for T units of time. A certain component
is essential for the system to be operative. When it fails it must be
replaced. However, there are n types of components that can be used;
some types are more costly but tend to last longer -- others are cheaper
but are not likely to last as long.

The general problem is to assign the initial component type and
all subsequent replacements so as to minimize the expected cost of pro-
viding an operative component for the   the system.

For example, a car owner who keeps his car for a random length of
time must from time to time replace the battery. A number of possible
replacement types exist. Longer lasting batteries cost more. However,
the trade-in price for the car will not reflect the type or condition of
battery in use. Thus, the problem of type of battery to use as a replace-
ment naturally presents itself if the car owner is concerned with operating

costs. It will be seen that making optimal replacement decisions may produce only modest savings for the individual car owner (or system operator). However, for the large fleet owner, such as the Federal or State government, the cumulative advantage of making optimal decisions may be very large.

Intuition would suggest that when the system is relatively new and a replacement is necessary it would be rational to use an expensive type providing the additional component life bought is worth the extra price; similarly when the system is old, an inexpensive type should be used. Consideration of this problem is intended to make this intuition more precise and to explore under what conditions it is, in fact, valid.

The problem can be viewed as one in discrete time parameter negative dynamic programming. The state of the system is its age when a replacement is necessary. The possible decisions are the different types of components to use. The costs are the costs of the types used. Laws of motion are determined by the life distribution functions of the different component types. The life of the system is T.

The paper by Strauch [3] covers foundations. An optimal policy exists; it will be non-randomized and stationary; the optimal value function will satisfy the optimality equation and any value function corresponding to a policy that satisfies the optimality equation will be an optimal value function.

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2. Fixed Horizon Case

We assume, here, that it is known precisely when the entire system will be terminated. Instead of using the age of the system, it is convenient to let t, the time until system termination, be the state of the system when a replacement is necessary.

We assume throughout that the distribution of life of a type i component is exponential with mean λ_i^{-1}. In most cases this is not a realistic assumption. However, as in the usual justification, we consider the assumption to be a first approximation.

Let C_i denote the cost of a type i replacement and $V(t)$ denote the expected cost of all remaining replacements given t units of time until termination and an optimal replacement policy is used. Assume that the types are indexed so that $\lambda_1 C_1 \geq \lambda_2 C_2 \geq \cdots \geq \lambda_n C_n$.

The main result that appears in [1] is as follows: $V(t)$ is piece-wise linear and concave. An optimal policy can be characterized as n intervals (some of which may be vacuous) where a type i component is used for replacement when t units remain until termination and t is in the ith interval from the origin.

The proof of the theorem exploits the optimality equation

$$(1) \qquad V(t) = \min_i \left\{ C_i + \int_0^t V(t-x) \, \lambda_i \, e^{-\lambda_i x} \, dx \right\} , \qquad t > 0 \ .$$

Once it is established that $V(t)$ is continuous at all $t > 0$, the expression $\psi_i(t)$, say, within the brackets of (1) is easily shown to be differentiable for each i. Thus, if i is optimal at t, on differentiation it is seen that

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$$\psi_i'(t) = \lambda_i c_i \ .$$

If i is uniquely optimal at t then from (1) and continuity it is optimal in the interval $(t, t+\epsilon)$ for some $\epsilon > 0$. If i and j are both optimal at t then from (1) and the above derivative it follows that within $(t, t+\epsilon)$, for some $\epsilon > 0$, the type with the smallest λC is optimal. The stated structure of the optimal policy follows.

An algorithm for obtaining the values of t where the optimal replacement type changes is a corollary of the optimal policy structure. The values are obtained recursively. Let π_k denote the optimal policy when only types $1, \ldots, k$ are available; π_1 is obvious. Let $V_k(t)$ be the value function using π_k. Since the optimal value function is continuous the value t^* satisfying

$$(2) \qquad V_k(t^*) = C_{k+1} + \int_0^{t^*} V_k(t^*-x) \ \lambda_{k+1} \ \exp(-\lambda_{k+1}x) \ dx$$

is that value of time until termination such that π_{k+1} uses type $k+1$ for all $t \geq t^*$; for $t \leq t^*$, π_{k+1} coincides with π_k. If $n = 2$, $C_1 < C_2$, $\lambda_1 C_1 > \lambda_2 C_2$,

$$t^* = \lambda_2^{-1} \ \log\left[\frac{C_1\lambda_1 - C_2\lambda_2}{C_1\lambda_1 - C_2\lambda_2}\right] \ .$$

The algorithm automatically excludes those types which would never be used; their corresponding intervals would be vacuous. If $C_j \geq C_i$, $\lambda_j > \lambda_i$ (or $C_j > C_i$, $\lambda_j \geq \lambda_i$), the case where a larger cost purchases

a shorter expected life, then type j is never used. This is shown formally in [1] by comparing the policy π_j of first using type j and replacing optimally thereafter with the policy π_i of first using type i and replacing optimally thereafter. If V_{π_j} and V_{π_i} are the respective value functions then for all $t > 0$,

$$(3) \qquad V_{\pi_j}(t) - V_{\pi_i}(t) = (C_j - C_i) + \int_0^t V(t-x)\left(\lambda_j \exp[-\lambda_j x] - \lambda_i \exp[-\lambda_i x]\right) dx.$$

The first term on the right of (3) is non-negative (positive). Because the exponential distribution with the greater mean dominates, and since $V(t-x)$ is decreasing in x the second term is also positive (non-negative). Thus $V_{\pi_j} - V_{\pi_i} > 0$ and it is never optimal to replace with type j.

The case of $n = 2$ suggests the conjecture made in [1] that a sufficient condition for type j never to be used is that $C_j > C_i$ and $C_j \lambda_j \geq C_i \lambda_i$ (or $C_j \geq C_i$ and $C_j \lambda_j > C_i \lambda_i$). The comparison (3) fails to be conclusive because the condition $\lambda_j > \lambda_i$ is not implied and a useful estimate of the integral in (3) is not available. Smith [2] uses an ingenious comparison to avoid the necessary evaluation of an integral involving the function V. Considering the case where $C_j \lambda_j > C_i \lambda_i$ but $\lambda_j < \lambda_i$ (hence, $C_j > C_i$). Smith compares π_j with $\tilde{\pi}_i$, a specified randomized policy. $\tilde{\pi}_i$ initially uses type i. At each replacement a chance experiment is carried out so that type i is used again with probability $(\lambda_i - \lambda_j)/\lambda_i$. When for the first time the chance experiment indicates that type i is not to be used then that replacement and all subsequent ones are made optimally. The usefulness of this device appears

in the fact that the length of time until a replacement other than type i is made has an exponential distribution with mean λ_j^{-1}, the same as would be the distribution of the length of time until the second component is introduced under policy π_j. Thus, in the comparison between V_{π_j} and $V_{\widetilde{\pi}_i}$ analogous to (3) the integral vanishes and

$$V_{\pi_j}(t) - V_{\widetilde{\pi}_i}(t) = (C_j - C_i) - M_i \, ,$$

where M_i is the expected cost of using type i components as a result of the chance experiments. M_i can be calculated, and it then follows that

$$V_{\pi_j}(t) - V_{\widetilde{\pi}_i}(t) \geq 0 \, , \qquad \text{for all } t > 0 \, .$$

Given the foregoing it is then possible to assert, without loss of generality, that the types are indexed so that $C_1 < C_2 < \cdots < C_n$ and $C_1\lambda_1 > C_2\lambda_2 > \cdots > C_n\lambda_n$, for otherwise some types could be excluded in advance. Given this ordering, it is easily argued that if t is small enough, type 1 should be used as a replacement. Thus, if no value of t^* satisfies (2), one can conclude inductively that type $k+1$ is never used and its interval of optimality is vacuous.

The optimal policy structure seems to reinforce intuition. However, it is not difficult to produce counter-examples, deviating from the assumption of exponential component life, where such a structure is not optimal. In fact, the exponential distributions may belong to a minority of

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distributions for which this structure holds. Whether the exponential assumption, nevertheless, represents a useful first approximation remains to be investigated.

3. Random Horizon Case

One serious limitation on the model considered to this point is the assumption that the entire system (of which our component of interest is a part) will be discarded at a fixed time. This assumption may hold when a strict disposal policy is in effect which specifies at what time the system will be abandoned. However, even when such a policy is in effect there is usually a provision for disposing of the system if it fails prior to the mandatory time of disposal. Within this context the fixed horizon model is extended in [1].

It is assumed that the system will be disposed at a fixed time or before if a failure occurs. The length of time until failure is exponentially distributed with mean α^{-1} (say). Thus the length of time until disposal is a truncated exponential distribution. Within this model it is still convenient to let the state of the system be designated as the maximum amount of time remaining until the system terminates. Analogous to (1), interpreting $V(t)$ as before, we have the optimality equation

$$(4) \quad V(t) = \min_i \left\{ C_i + \int_0^t e^{-\alpha x} V(t-x) \, \lambda_i \, e^{-\lambda_i x} \, dx \right\} , \quad t > 0 .$$

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Employing the same chain of reasoning as in the fixed horizon case, the expression, $\psi_i(t)$, within the brackets of (4) has a derivative which is given by

$$\psi_i'(t) = (\alpha + \lambda_i)C_i - \alpha V(t) \quad ,$$

if i is optimal at t. The interval structure of an optimal policy still holds. V is still a concave function. However, the piecewise linearity gives way to the appropriate solution of the differential equation

$$V'(t) = (\alpha + \lambda_i)C_i - \alpha V(t)$$

over the interval for which type i is the optimal replacement.

It should be remarked that this model is mathematically identical to the fixed horizon case with discounted cost criterion. Hence, the interval structure for an optimal policy holds when costs are discounted.

It is to be expected that the assumptions regarding the distribution of T, the system life, can be further weakened. To this end suppose F is the distribution function of T. It is necessary, now, to define the state of the system as, s, the length of time the system has been operating, rather than the time remaining until termination. Let $U(s)$ denote the expected future cost of supplying the needed component given the system has been operating for s units of time, a replacement is necessary,

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and all replacements will be made optimally. The optimality equation that $U(s)$ satisfies is

$$(5) \qquad U(s) = \min_i \left\{ C_i + \int_0^\infty \lambda_i \, e^{-\lambda_i x} \, U(s+x) \, \frac{1 - F(s+x)}{1 - F(s)} \, dx \right\}$$

$$= \min_i \{\varphi_i(s)\} \ ,$$

where φ_i $(i = 1, \ldots, n)$ is the expression within the brackets of (5).

Some limitations on F must be imposed if the same interval structure for an optimal policy as obtained for the fixed horizon case is to prevail. One can easily construct examples of F where the optimal policy will not have the interval structure. However, if F has an increasing failure rate (IFR) the property should hold. Below is a sketch of the proof that this is, indeed, true for the case where $n = 2$ and F is strictly IFR. In addition to F being IFR it is also assumed that F is continuous except, perhaps, at x_0 where $F(x_0) = 1$.

As before, it can, without loss of generality, be assumed that $C_1 \lambda_1 > C_2 \lambda_2$, $C_1 < C_2$. Using the assumption of continuity it is straightforward to prove that $U(s)$ is continuous at all $s < x_0$. Then $\varphi_i(s)$, $i = 1, 2$ is differentiable and if i is optimal at s

$$\varphi_i'(s) = r(s) \, U(s) - (\lambda_i + r(s)) C_i \ ,$$

where $r(s)$ is the failure rate function of F. Since r is increasing there is a value $\bar{s}$ such that either

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$$r(\bar{s}) = \frac{c_1\lambda_1 - c_2\lambda_2}{c_2 - c_1}$$

$$= \bar{r} \qquad\qquad (\text{say})$$

or $r(s) < \bar{r}$ or $r(s) > \bar{r}$ for all $s \geq 0$. If $r(s) < \bar{r}$ for all $s \geq 0$ take $\bar{s} = \infty$; if $r(s) > \bar{r}$ for all $s \geq 0$ take $\bar{s} = 0$. Now if both types 1 and 2 are optimal at system age s then

$$(6) \qquad\qquad \varphi_2'(s) \geq \varphi_1'(s) , \qquad\qquad \text{if } s < \bar{s}$$

$$\leq \varphi_1'(s) , \qquad\qquad \text{if } s > \bar{s} .$$

By continuity, if s^* is an age where the optimal replacement type switches, then both types are optimal at s^*. However, by (6) in the interval $s < \bar{s}$ only a switch from type 2 to type 1 is possible and in the interval $s > \bar{s}$ only a switch from type 1 to type 2 is possible. In the latter case type 2 would be optimal for all system ages $s > s^*$ where $s^* > \bar{s}$ and s^* is the age at which the switch from type 1 to type 2 takes place. However, a direct comparison with the policy of always using type 1 for all $s > s^*$ shows that always replacing with type 2 for $s > s^*$ cannot be optimal. Hence, a switch from type 1 to type 2 for any $s > \bar{s}$ cannot be optimal. The remaining possibilities are consistent with what was to be proved.

Recently, Derman and Smith have extended this result to $n > 2$. An algorithm is also suggested for obtaining switching points.

Bounds

An upper bound on $U(s)$ is easily obtained by evaluating the policy that always uses type n as a replacement. Thus

$$U(s) \leq \lambda_n C_n \ E(T | T > s) + C_n \quad .$$

To obtain a lower bound a modified problem is considered in [1]. The modification consists in allowing a rebate of the amount $C_j - C_1$ if a type j component is in use at the time the system terminates. The expected optimal cost for this problem is no larger than the expected optimal cost for the original problem. However, for the modified problem it is optimal to use a type n component for every replacement (assuming $\lambda_n C_n < \cdots < \lambda_1 C_1$). Thus, a lower bound for $U(s)$ is

$$U(s) \geq \lambda_n C_n \ E(T | T > s) + C_1 \quad .$$

If $C_n - C_1$ is small the question of optimal replacement has little practical interest unless many systems are in operation so that the optimality pay-off is manifested in the aggregate.

Problem Variations

Besides the modified problem alluded to in the previous section several other modifications are considered in [1] for the fixed horizon case.

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One variation assumes that there are two replacement types. An infinite number of type 2 components are available; however, only one type 1 component is available. It is shown that the same two interval structure is optimal. The switching point is evaluated but is not the same as the original $n = 2$ problem. Optimal stopping theory is the method used to establish the result. The method, although not the switching point, is free of the exponential component life assumption allowing one to obtain the interval structure for an optimal policy under weaker assumptions.

The reverse situation where an infinite number of type 1 components are available and only one of type 2 also has the same optimal policy structure. The switching point, as would be expected, is the same as in the original problem. The method, here, is to guess that the optimal policy is of this form and verify that value function satisfies the optimality equation.

The problem with the rebate that yields the lower bound discussed in the previous section assumes, except for at least one type, that there are only a finite number of replacements. That is, only one type must have an infinite supply in order that a replacement is always available. The horizon can be of random length as well as fixed. The optimal policy is to always use the type available that has the smallest λC.

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The problem can be viewed as one in discrete time parameter negative dynamic programming. The state of the system is its age when a replacement is necessary. The possible decisions are the different types of components to use. The costs are the costs of the types used. Laws of motion are determined by the life distribution functions of the different component types. The life of the system is T.

This problem was originally considered by the authors in "A Renewal Decision Problem". This paper summarizes these results and considers various extensions.